

Statistical properties of spectra of pseudointegrable systems

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We analyze the spectral properties of various quantum pseudointegrable billiards (rhombus, gnomon, deltoid) and link them to the genus of the invariant surface of the corresponding classical model. Numerical investigations of the quantum billiards are completed by an experimental study of microwave resonators. Absorption spectra of microwaves in “L-shaped” resonators are measured and the distributions of eigenfrequencies are investigated.

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I. INTRODUCTION

Classical billiards are not only used for entertainment but provide mathematicians and physicists with an interesting realization of dynamical systems. The dynamics of a particle bouncing on a plane between hard walls depends in a characteristic way on the shape of the billiard. Its motion in a rectangular or circular billiard is regular and the system is integrable. However, if a circular obstacle is put inside the rectangle (Sinai billiard) or if the circle is transformed into a stadium (Bunimovich billiard) the system becomes chaotic [1,2].

During the last decade quantum analogs of classical billiards have also been studied. The problem is equivalent to solving a stationary Schrödinger equation in a two-dimensional potential well of infinite depth of the shape of the billiard enclosure. The ergodicity of the classical model manifests itself in the statistical properties of the spectrum of the corresponding quantum system. Whereas for integrable systems the Poisson distribution of the level spacing statistics was found [3–5], the spectra of the classically chaotic systems display level repulsion and the Wigner statistics of level spacings [4,6–10]. Spectral fluctuations of the classically chaotic systems are well described [11] by ensembles of random matrices. Depending on the symmetry properties of the system the orthogonal, unitary, or symplectic ensemble should be used [12].

The difference between the integrable and chaotic classical systems can be easily seen if one investigates the behavior of their trajectories in the phase space. The trajectories of an integrable system are confined to an invariant manifold which is topologically equivalent to a torus (an object with genus g equal to unity), while the trajectories of a chaotic system explore the whole available phase space. There are also, however, dynamical systems the phase space trajectories of which are bounded on invariant surfaces which are topologically equivalent to multihanded spheres (objects with genus $2 \leq g < \infty$). These systems corresponding to finite genus are called *pseudointegrable* [13–17].

From the quantum point of view the difference between the integrable and chaotic systems can be described as follows: In the integrable case, because of the existence of the classical invariant torus, a semiclassical quantization is possible [Einstein-Brillouin-Keller (EBK) method] [18]. For the integrable billiards this procedure leads to quantum levels which exhibit level spacing statistics according to the Poisson distribution. The chaotic case is more complicated. The absence of an invariant surface makes an EBK quantization impossible and the quantum levels can be obtained by solving the corresponding Schrödinger equation. A formulation of semiclassical quantization conditions based on the classical periodic orbits [19,20] has been improved only recently by a novel technique of resumming its divergent series [21–23].

Pseudointegrable systems feature two contradicting properties, at first glance. From the classical point of view these systems are not chaotic since they possess an invariant surface. The complicated topological structure of this surface makes, however, the EBK quantization questionable. In addition, it is not clear at all to what extent the methods of periodic orbit quantization might be suitable for this case [24,25]. On the other hand, the level spacing distribution of pseudointegrable quantum systems is closer to the Wigner distribution, characteristic of classically chaotic systems, than to the Poisson distribution.

Quantized versions of pseudointegrable billiards were studied first by Richens and Berry [14]. They found a level repulsion in spectrum of an L-shaped billiard, typical of classically chaotic systems. Recent numerical studies of a rhombus billiard [26–28] have confirmed this fact and have shown that the results obtained for pseudointegrable systems differ significantly from the Wigner distribution. A nonzero probability of level clustering was found for one symmetry class of a rhombus billiard by Biswas and co-workers [26,27], but this observation was based on rather small statistics containing less than 350 levels each. These results have not been observed by Shudo and Shimizu [28], who obtained for the same quantum

system a statistics containing about 3000 levels and reported a level repulsion.

An interesting work on pseudointegrable systems is credited to Cheon and Cohen [29], who analyzed an approximation of the Sinai billiard constructed of N rectangles. For a large number N they observed that the level statistics become closer to the Wigner distribution. Since this pseudointegrable system is characterized by a genus equal to N , one might speculate that in the limit $g \rightarrow \infty$ the level spacing distribution tends to the Gaussian-orthogonal-ensemble (GOE) –like distribution, characteristic of the classical chaotic systems possessing a generalized time-reversal symmetry [11]. However, in a recent study of a billiard rhombus with the vertex angle not commensurate with π (characterized by an infinite genus), a discrepancy with respect to GOE distribution was reported [28].

Pseudointegrable systems may also be constructed by putting a singular, pointlike perturbation into a regular billiard [30]. On the one hand, it is difficult to find a rigorous definition of the classical limit of this model (the same holds true for its modified version [31]); on the other hand, the quantum calculations are usually easier than for polygonal billiards [32]. Shigehara *et al.* have computed 10^3 eigenvalues of this system and reported a linear repulsion [33]. The obtained distribution of the level spacings might be well approximated [34] by an ensemble of additive random matrices [35–37]. The same ensemble has been applied to explain preliminary results obtained for an L-shaped billiard [38]. In spite of numerous papers on the pseudointegrable quantum systems recently published, several questions still remain open. It is not at all clear whether the spectral properties of these quantum systems depend on the energy, whether they are uniquely determined by the genus g of the corresponding classical phase space, or what the properties are of a classically nonchaotic system with $g \rightarrow \infty$.

The aim of this paper is to present results concerning the properties of pseudointegrable quantum billiards and corresponding microwave cavities studied experimentally. Since authors of previous papers on this subject have presented different, and often contradicting, opinions concerning the properties of pseudointegrable quantum billiards, we would like to verify some statements and formulate more general hypotheses. To this end, we analyze several different quantum billiards described by a free parameter, a change of which allows one to obtain a better statistics.

For a numerical investigation we have selected three different shapes of the billiard presented in Fig. 1: rhombus, gnomon (“L-shaped” billiard), and deltoid (“kite”). The genus of the invariant manifold in the phase space of a rhombus depends on the vertex angle φ and is finite for any angle commensurate with π . Choosing an appropriate angle of the rhombus, one may find several realizations of a simplest pseudointegrable system, characterized by $g = 2$. The two other models are even more suitable for our purposes, since both enable a construction of a one-parameter family of systems with a constant value of the genus g . The symmetrical L-shaped billiard might be described by the length of its arm a ,

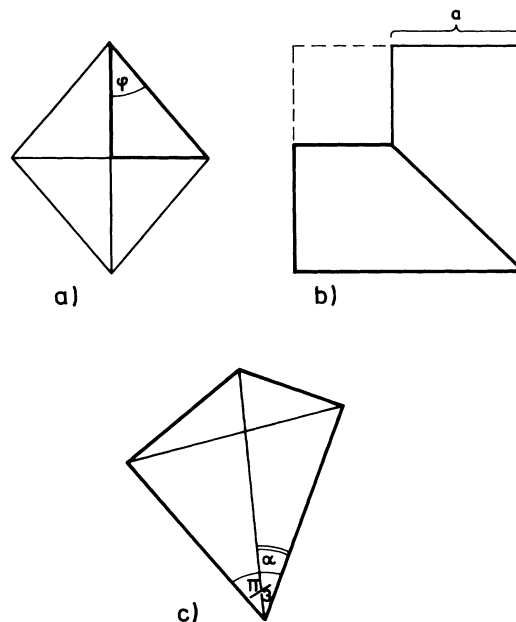


FIG. 1. Pseudointegrable billiards of genus 2 studied: (a) rhombus, (b) gnomon, (c) deltoid.

while the asymmetrical deltoid is parametrized by the angle α , as can be seen in Fig. 1. Classical properties of pseudointegrable billiards are briefly reviewed in Sec. II of this work.

The character of a quantum spectrum can be described by the level spacing distribution $P(s)$, which characterizes the correlations between adjacent levels, and by the spectral rigidity $\Delta_3(L)$, which describes the long-range correlations of the spectrum [10]. In Sec. III we present an analysis of the dependence of both these quantities on the energy and on the parameters φ , a , and α for each system, respectively.

A problem of describing a quantum particle bouncing in a given two-dimensional billiard is closely related to the classical system consisting of electromagnetic waves propagating in a flat resonator having the shape of the billiard. Similarity between the Schrödinger equation, describing the quantum billiard, and the Helmholtz wave equation justifies a direct comparison of the results obtained from microwave experiments [39–43] with the quantum theory. The distribution of spacings between adjacent eigenfrequencies of the resonator constructed in a shape of the Sinai billiard agrees with the Wigner surmise [39], but some deviations due to the bouncing-ball mode have been reported [43]. The spacing distribution of a rectangular microwave resonator, corresponding to an integrable billiard, does not confer to the Poisson distribution [34]. Observed discrepancies can be explained by the presence of a wire antenna used for transmitting the microwaves to the system. This part of the experimental apparatus plays the role of a point perturbation, and the entire system can be associated with a singular pseudointegrable billiard [30,32].

In Sec. IV we describe results of a microwave experiment performed with the cavity in the shape of asym-

metric L-shaped billiards [44]. Note that due to the presence of the antenna in the cavity one should not expect a coincidence of the measured eigenfrequencies with the numerically computed levels of the corresponding quantum billiard. On the other hand, a comparison of the results obtained for rectangular and L-shaped resonators provides arguments supporting the observations obtained by a numerical study of pseudointegrable quantum billiards. We conclude in Sec. V, summarizing our results and presenting a list of open problems.

II. CLASSICAL PSEUDOINTEGRABLE BILLIARDS

Let us focus our attention on the billiards with a polygonal enclosure. A polygon billiard is called rational [13] if all angles between sides of the polygon P are rational multiples of π . The phase-space trajectories are then confined to an invariant manifold R which is obtained by unfolding the original billiard and identifying the opposite sides. Let P be a rational polygon with vertex angles $\pi n_i/m_i$, $i = 1, \dots, k$ and let M be the least common multiple of m_i . The genus g of the corresponding object R is then [14–16]

$$g = 1 + \frac{M}{2} \sum_{i=1}^{i=k} \frac{n_i - 1}{m_i}. \quad (1)$$

One immediately sees that g equals unity for billiards having the shape of a rectangle, equilateral triangle, or of triangles with vertex angles equal to $\pi/3, \pi/2, \pi/6$ and to $\pi/2, \pi/4, \pi/4$. Polygon R has in these cases the shape of a rectangle, parallelogram, or regular hexagon, respectively, and is (after identification of the opposite sides) topologically equivalent to a torus. These billiards are integrable [45]. It is worth noting at this point that only for these polygonal billiards one can solve exactly the corresponding Helmholtz equation [46,47].

The simplest pseudointegrable billiards are characterized by the genus g equal to 2. A billiard with an L-shaped enclosure belongs to this class. Many other examples are known: a rhombus with vertex angle equal to $\pi/3$, or, in general, a $\pi/3$ parallelogram, deltoids with angles $(\pi/3, \pi/2, 2\pi/3, \pi/2)$ and $(\pi/4, \pi/2, 3\pi/4, \pi/4)$, or isosceles triangles $(\pi/6, 2\pi/6, \pi/6)$ and $(\pi/5, 3\pi/5, \pi/5)$.

The genus 3 is characteristic for the $\pi/4$ parallelogram or for a billiard containing three rectangular steps (an additional “step” added to the gnomon). Moreover, in accordance with Eq. (1), the rhombus with vertex angle equal to $\pi/(N+1)$ or the N -step rectangular staircase correspond to manifolds of genus $g = N$. Examples of billiards corresponding to each finite genus are therefore known.

The pseudointegrable billiards are not ergodic since the phase-space trajectories of those systems are confined on two-dimensional invariant manifolds. The existence of such a manifold indicates the existence of an additive integral of motion J (besides the Hamiltonian). However, it is difficult to distinguish empirically the classical trajectory in the pseudointegrable billiards from that in chaotic billiards with a positive Lyapunov exponent [48].

In the simplest case the integral of motion J is in an involution with the Hamiltonian H ,

$$\{H, J\} = 0, \quad (2)$$

where $\{ \}$ denotes the Poisson bracket. The topological theorem of Arnold [49] then implies that the invariant surface is equivalent to a two-dimensional torus (the system is integrable). In the general case of a rational polygon, however, the integral of motion J does not fulfill the involution equation (2). Instead of this, the Poisson bracket $\{H, J\}$ is proportional to a sequence of δ functions which are localized on the “bad” vertices of the polygon. At those points the geodesic flow of the billiard becomes singular. As a consequence, the topological theorem does not apply and the invariant surface has a more complicated topological structure. In general the bad vertices enhance the genus of the invariant surface as described by formula (1).

From the quantum point of view the existence of the singularities in the classical geodesical flow makes EBK semiclassical quantization impossible. Moreover, the overall structure of these singularities (they are localized at isolated points) explain the similarity of the level statistics obtained for pseudointegrable polygonal billiards and for billiards with singular perturbations [30,32].

III. QUANTUM PSEUDOINTEGRABLE BILLIARDS

A quantum billiard is a simple quantum-mechanical model consisting of a point particle moving in the two-dimensional infinite potential well

$$V(x, y) = \begin{cases} 0, & \{x, y\} \in \Omega \\ \infty, & \{x, y\} \notin \Omega, \end{cases} \quad (3)$$

where the compact set Ω is determined by the shape of the classical billiard $\partial\Omega$. Analysis of the quantum model is based on solving the stationary Schrödinger equation, which in this case reduces to the Helmholtz equation

$$(\nabla^2 + k^2)\psi(x, y) = 0, \quad (4)$$

with the Dirichlet boundary condition $\psi(x, y) = 0|_{\{x, y\} \in \partial\Omega}$.

The above equation may be solved analytically for integrable billiards only [46,47]. Studying pseudointegrable billiards, we have to rely on numerical solutions of Eq. (4), which were performed according to the numerical technique described by Riedel [50] and applied in other papers [4,14,46].

Let $\{E_0, E_1, E_2, \dots\}$ denote the infinite sequence of the eigenvalues and $\{\phi_0, \phi_1, \phi_2, \dots\}$ the corresponding sequence of the eigenfunctions of Eq. (4). According to the Weyl formula [51], the mean spacing between levels of the quantum billiard $\bar{s} = \langle E_{i+1} - E_i \rangle$ is in the first approximation inversely proportional to the area of the billiard Ω . Further corrections to this formula take into account the specific features of the billiards enclosure, like corners and edges [51]. We have used the Weyl formula to check the accuracy of numerical computations and to estimate the percentage of eigenvalues missing. For each

quantum system, we obtained about 3000 levels, and the error of less than 2% did not affect seriously the statistical properties of the spectra described by the level spacing distribution $P(s)$ and the spectral rigidity $\Delta_3(L)$ [12].

We solved numerically the Helmholtz equation (4) for pseudointegrable polygonal billiards presented in Fig. 1: rhombus, gnomon, and deltoid. Statistical properties of the quantum spectra should be studied separately for each parity class. For a rhombus billiard with the vertex angle 2φ the odd-odd class was considered, which is equivalent to the right triangle billiard with the vertex angle φ . We have chosen billiards defined by angles $\varphi_{21}=\pi/5$, $\varphi_{22}=\pi/10$, and $\varphi_{23}=\pi/8$, which are characterized by genus $g=2$. The same genus corresponds to the L-shaped billiards. We studied the odd-parity class of symmetric billiards constructed from the unity square by excluding a corner square of length $a < 1$ (see Fig. 1) and took $a=0.1, 0.2, 0.3, 0.4, 0.5$. As examples of another class of genus 2 systems, we examined “kite” billiards defined by irrational angles $\alpha_1=\pi(\sqrt{2}-1)/4$, $\alpha_2=\pi(\sqrt{3}-1)/4$, and $\alpha_3=\pi(\sqrt{5}-1)/2$. Right triangle billiards defined by angles $\varphi_{31}=\pi/14$, $\varphi_{32}=\pi/7$, $\varphi_{33}=3\pi/14$, and $\varphi_{34}=\pi/12$ were analyzed as representatives, of systems of genus 3.

Careful analysis of the data obtained for several systems belonging to different classes confirms the following hypothesis: (i) statistical properties of spectra of quantized pseudointegrable systems depend on the energy only very weakly, (ii) spacing distribution exhibits level repulsion, (iii) level spacing distributions characteristic of all genus 2 systems are neither Poisson nor Wigner, and (iv) statistical properties of systems of higher genus are closer to those predicted by random matrices and characteristic of classically chaotic billiards. In order to describe obtained histograms of level spacing $P(s)$ in a quantitative way, it is natural to fit data to distributions interpolating between Poisson distribution $P_P(s)=e^{-s}$ and the Wigner surmise

$$P_W(s)=s\frac{\pi}{2}e^{-s^2\pi/4}. \quad (5)$$

The distribution

$$P_q(s)=(1+q)C(q)s^q\exp[-C(q)s^{1+q}], \quad (6)$$

with $C(q)=\{\Gamma[(2+q)/(1+q)]\}^{1+q}$, was proposed by Brody [52] without any physical arguments, but seems to describe quantitatively spectral properties of several quantum systems. Another phenomenological distribution has recently been proposed by Izrailev [53,54],

$$P_\beta(s)=As^\beta(1+sB\beta)^{f(\beta)}\exp\left[\frac{-\pi^2\beta s^2}{16}-\frac{\pi}{4}(2-\beta)s\right], \quad (7)$$

where

$$f(\beta)=\frac{2^\beta(1-\beta/2)}{\beta}-0.16874,$$

and the constants A and B are determined by the normalization conditions. The usefulness of the above distribu-

tion has been demonstrated on the model of band random matrices [55]. Another distribution based on the model of additive random matrices has been proposed by Lenz and Haake [36],

$$P_\lambda(s)=\left[\frac{su(\lambda)^2}{\lambda}\right]\exp\left[\frac{-u(\lambda)^2s^2}{4\lambda^2}\right]\times\int_0^\infty e^{(-\xi^2-2\xi\lambda)}I_0\left[\frac{s\xi u(\lambda)}{\lambda}\right]d\xi, \quad (8)$$

where $u(\lambda)=\sqrt{\pi}U(-1/2,0,\lambda^2)$, $I_0(x)$ stands for the modified Bessel function, and $U(a,c,x)$ denotes the Tricomi function [56]. This formula was rigorously derived for 2×2 random matrices, but gives a good approximation to the statistics for arbitrary large matrices [37]. For $q=\beta=\lambda=0$ all three above formulas reduce to the Poisson distribution while the GOE distribution is approximated for $q=\beta=1$ or in the limit $\lambda\rightarrow\infty$.

The distribution of Berry and Robnik [57] is obviously not adequate for our case, since [because of $P(0)>0$] it does not portray the observed feature of level repulsion (iii). This property is guaranteed by the distributions (6)–(8). In order to measure the quality of the fit we used the χ^2 test for each of the above distributions. Working with about 3000 levels for each system we made histograms consisting of 30 bins. Following the standard procedure we have not taken into account bins for which the number of events expected according to the distribution analyzed is smaller than 7, so the number of degrees of freedom ν is equal to 27, 28, or 29, depending on the set of numerical data and the distribution. The quality of the fit is characterized by the normalized sum of squared deviations χ_ν^2 and the confidence level $c=\log_{10}[Q(\chi_\nu^2)]$. Parameters characterizing the level spacing distribution of spectra of pseudointegrable billiards are collected in Table I.

Results of the fitting procedure demonstrate that none of the above distributions passes the χ^2 test satisfactorily (confidence level of order 10^{-4} – 10^{-13}). On the other hand, each distribution (6)–(8) gives an approximation to the results obtained for pseudointegrable quantum billiards, and the values of the best fit might be applied to characterize the statistics quantitatively. The simple Brody distribution (6) often gives a better fit than the more sophisticated distribution (7), while in some cases the best fit is provided by the random-matrix distribution (8). Since this distribution was definitely the best in describing spectra of the billiard with a point interaction [32,34], we have chosen it to characterize the obtained distributions by the parameter λ .

The relatively low confidence level Q , which characterizes the quality of the fits, influences the accuracy of the value of the parameter λ obtained in fitting. Observe that the genus 3 systems are described in higher values of the parameter λ than the systems characterized by genus 2, and are therefore closer to the GOE distribution. The only exception is the $\pi/12$ rhombus. All the values of the fitting parameters obtained for this billiard (with a significantly smaller confidence level) are much lower than those characteristic of other genus 3 systems. This

TABLE I. Parameters characterizing spectra of quantum pseudointegrable billiards: (a) quarter of rhombus with vertex 2φ ; (b) $\pi/3$ deltoids with angle α ; and (c) “L-shaped” billiards parametrized by the side length a . Best-fit values and the levels of confidence are presented for the Lenz-Haake (8), Brody (6), and Izrailev (7) distributions.

System parameter	Genus g	Lenz Haake		Brody		Izrailev	
		λ	$e(q)$	q	$c(\lambda)$	b	$c(b)$
$\varphi = \pi/10$	2	0.62	-3.8	0.72	-3.2	0.70	-4.0
$\varphi = \pi/8$	2	0.53	-3.5	0.67	-4.5	0.65	-4.5
$\varphi = \pi/5$	2	0.57	-3.7	0.70	-4.2	0.68	-4.8
$\varphi = \pi/14$	3	0.75	-3.6	0.78	-2.0	0.78	-2.1
$\varphi = \pi/12$	3	0.43	-11.5	0.57	-4.2	0.53	-5.2
$\varphi = \pi/7$	3	0.81	-1.7	0.80	-0.8	0.80	-0.9
$\varphi = 3\pi/14$	3	0.66	-1.6	0.73	-1.5	0.71	-1.7
$\alpha = \sqrt{2} - 1$	2	0.47	-5.5	0.60	-1.0	0.57	-2.1
$\alpha = \sqrt{3} - 1$	2	0.41	-5.0	0.59	-1.2	0.57	-1.9
$\alpha = 2\sqrt{5} - 2$	2	0.41	-5.2	0.55	-4.9	0.51	-7.2
$a = 0.1$	2	0.41	-5.9	0.54	-11.1	0.51	-12.5
$a = 0.2$	2	0.51	-4.6	0.65	-5.9	0.62	-7.4
$a = 0.3$	2	0.53	-6.1	0.62	-9.3	0.59	-10.8
$a = 0.4$	2	0.51	-2.7	0.63	-4.9	0.61	-7.2
$a = 0.5$	2	0.39	-12.5	0.48	-10.1	0.42	-10.7

peculiar behavior of this system has already been reported in [28], but still awaits an explanation.

In addition, the spectra of various genus 2 systems are not described by the same values of the fitting parameters. It is not yet clear whether these differences are statistically significant and denote different spectral properties of systems with the same genus, or alternatively, whether they are caused by not sufficiently large statistics of levels or by the lack of an exact interpolating formula. Though we have demonstrated that the genus of the classical phase space is an important factor influencing the statistical properties of the quantized systems, it is not possible to conclude at this stage that for each genus universal distributions of eigenlevels exist. Exemplary statistics of quantum billiards characterized by genus 2 and 3 are presented in Fig. 2.

Complementary information concerning the statistical properties of spectra might be obtained by a study of the spectral rigidity $\Delta_3(L)$. This quantity measures the correlation of the spectrum at the distance L . It is known that for the Poissonian spectra rigidity increases linearly, $\Delta_3 = L/15$, while for the GOE a logarithmic law (valid for $L > 1$) has been found [12],

$$\Delta_3(L) = \frac{1}{\pi^2} \ln(L) - \gamma, \quad (9)$$

where the constant $\gamma \approx 0.00695$. Figure 3 presents the spectral rigidity for a pseudointegrable billiard, while the dashed line denotes the Poisson-like spectrum and the solid line the GOE-type spectrum (9). Note that the rigidity $\Delta_3(L)$ is similar to all the systems of genus 2 and 3, respectively, at least for L of the order of 20. For larger values of L the properties of the corresponding classical billiards, including the variety of the system-specific periodic orbits, influence the fluctuations of the spectra [58], and the spectral rigidity of different systems with the same genus begins to differ. Spectra of genus 3 sys-

tems are more rigid than those obtained for $g=2$, and thus closer to the formula (9). On the other hand, in the whole range of L there exists a significant difference between the rigidity of the pseudointegrable systems and those of GOE, typical of classically chaotic systems. This observation does not contradict recent results of Biswas and Sinha [25], who tried to show a link between the genus of the classical phase space and long-range correlations of the quantum spectra.

IV. MICROWAVE EXPERIMENT IN AN L-SHAPED RESONATOR

Measuring the reflection spectra of microwave cavities yields information about the spectrum of the corresponding quantum billiard [39]. The presence of the antenna in the experimental apparatus [34] and the finite resolution of the measurement limit the obtained accuracy, but it is relatively easy to get larger numbers of eigenfrequencies than in the case of time-consuming numerical calculations involving diagonalizations of large matrices. The microwave resonator might be thus treated as an analog computer providing an alternative way to get complementary information on the corresponding quantum system. We have decided, therefore, to accomplish microwave experiments using cavities of the shape of pseudointegrable billiards.

The experimental apparatus used is analogous to that described in Ref. [34]. The system consists of an L-shaped resonator coupled to the power source by a microwave cable (see Fig. 5). The microwaves were excited by a thin wire antenna perpendicular to the top and bottom face of the cavity, these faces being parallel to each other with a distance of $d=8$ mm. The reflection of the microwaves was measured as a function of frequency. Each reflection minimum obtained corresponded to an eigenfrequency of the resonator. In order to increase the

amount of available data we repeated measurements for different values of the sides of the resonator a and b (see Fig. 4). The lengths a and b were in the range from 80 to 280 mm. The position of the antenna was fixed for each resonator ($x_a = 30, y_a = 80, \dots, 120$ mm).

The frequency of microwaves ν was scanned in the range 0.5–18 GHz, and depending on the area of the resonator, each spectrum contained 200 to 600 eigenfrequencies. A comparison with the Weyl formula showed that the total experimental loss was of the order of 25%. This rather large loss had two causes. A correlated loss results from the fact that the finite resolution of the experimental limits the smallest resolvable distance to several MHz. Second, a resonance cannot be excited if the position of the antenna coincides with a node line. This leads to an uncorrelated loss. From the fact that in Sinai billiards (where because of the level repulsion the corre-

lated loss can be neglected) the observed losses were between 5% and 10%, we conclude that about $\frac{2}{3}$ of the loss is correlated and $\frac{1}{3}$ is uncorrelated. To estimate the influence of the loss on the λ parameter, we calculated spectra of rectangular billiards and removed 25% of the eigenvalues, $\frac{2}{3}$ correlated and $\frac{1}{3}$ uncorrelated. The resulting λ values were of the order of 0.2 and 0.3. This suggests that, e.g., for our rectangular billiards in the frequency region below 10 GHz [34] the λ values are clearly affected by the loss, whereas in the higher frequencies for the rectangular billiards and probably for all frequencies for the L-shaped billiards the λ values should be only slightly influenced by the loss.

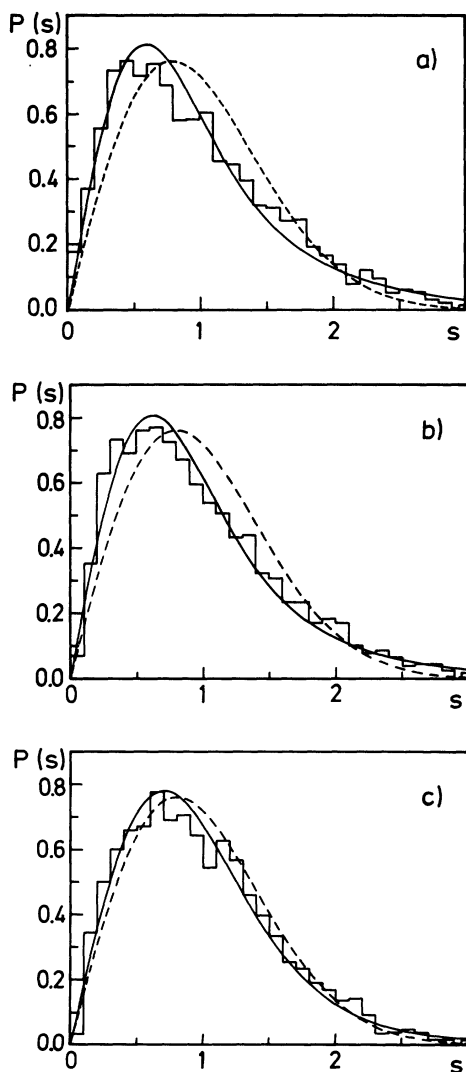


FIG. 2. Level spacing distribution evaluated numerically for genus 2 billiards: (a) deltooid with $\alpha = \sqrt{2} - 1$, $\lambda = 0.47$; (b) L-shaped billiard with $a = 0.4$, $\lambda = 0.51$ and a genus 3 billiard; and (c) a quarter of a $\varphi = \pi/14$ rhombus, $\lambda = 0.75$. The dotted curve represents the Wigner surmise.

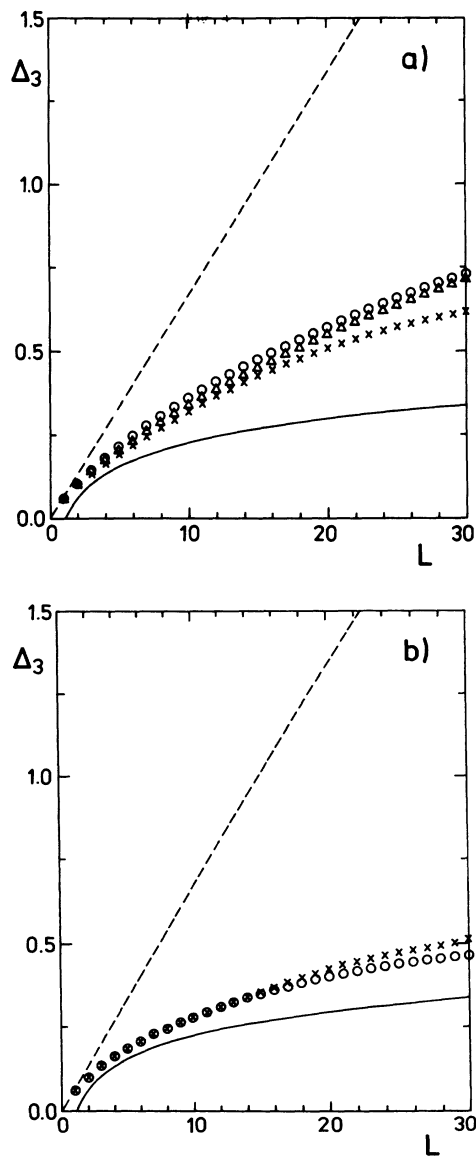


FIG. 3. Spectral rigidity obtained numerically for (a) genus 2 billiards, deltooid with $\alpha = \sqrt{2} - 1$ (\circ), L-shaped billiard with $a = 0.2$ (Δ), quarter of a $\varphi = \pi/5$ rhombus (\times); and (b) genus 3 billiards, quarter of a $\varphi = \pi/14$ (\circ) and $\varphi = \pi/7$ (\times) rhombus. Dashed line represents the Poisson spectrum and the solid line the GOE-like spectrum.

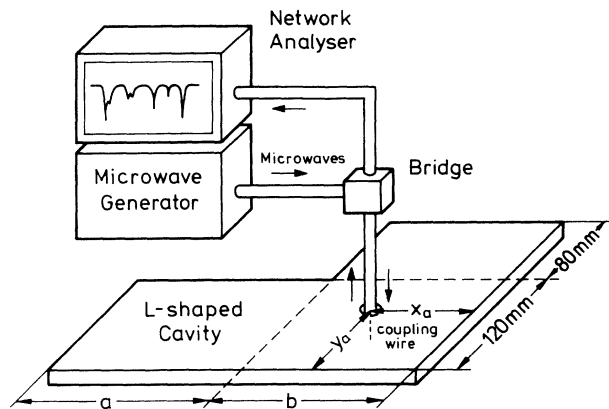


FIG. 4. Scheme of the experimental apparatus. The microwaves are irradiated via a copper wire (diameter $d=0.5$ mm) into an L-shaped cavity; (x_a, y_a) describes the position of this wire. The lengths of the sides a and b were varied. (For dimensions, see text.) The reflected microwave power is registered by means of a scalar network analyzer.

In order to analyze the spacing distribution of measured frequencies, we applied the standard technique of spectral unfolding [52] separately for each resonator. Since the coupling of the antenna with the system grows with frequency [34], we collected the data separately for different energy ranges. Figure 5 presents the experimental data obtained (a) for $5 \text{ GHz} < \nu < 10 \text{ GHz}$ and (b) for $15 \text{ GHz} < \nu < 18 \text{ GHz}$. The best fits with distribution (8)

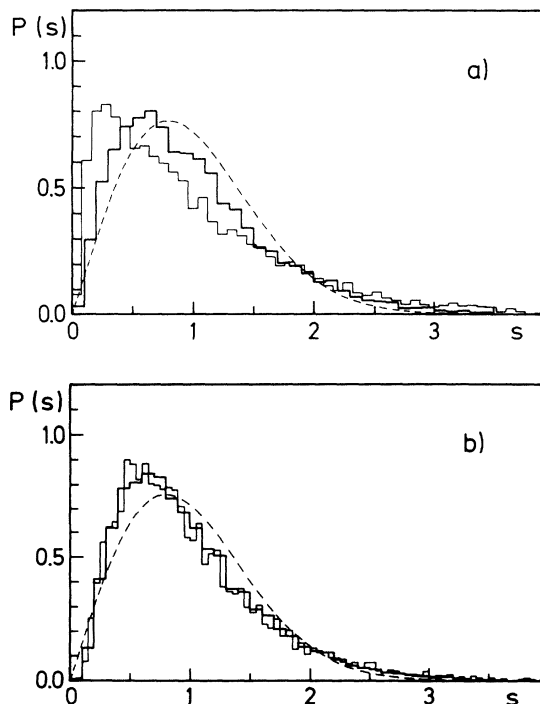


FIG. 5. Comparison of the level spacing distribution obtained experimentally for L-shaped resonators (bold lines) and rectangular resonators (faint lines) and the frequency range (a) $5 \text{ GHz} < \nu < 10 \text{ GHz}$, and (b) $15 \text{ GHz} < \nu < 18 \text{ GHz}$.

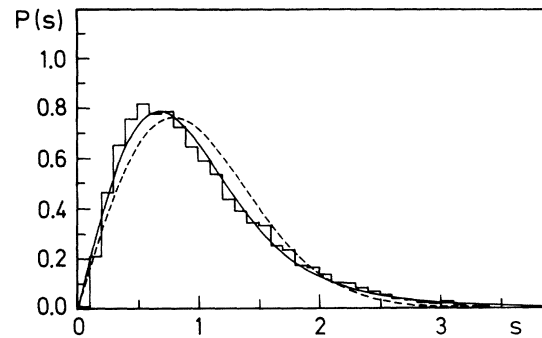


FIG. 6. Level spacing distribution collected from 102 L-shaped resonators of different sizes in the frequency range (b) $10 \text{ GHz} < \nu < 14 \text{ GHz}$. Solid line represents the distribution (8) with $\lambda=0.68$ and the dashed line the Wigner distribution.

are represented by bold lines. In order to facilitate a direct comparison, the results received for the rectangular resonators [34] are superimposed in the figure and represented by faint lines. As expected, the results obtained for pseudointegrable L-shaped resonators are closer to the Wigner distribution. The difference between the two cases studied is evident for small frequencies [see Fig. 5(a)] and becomes less significant for larger frequency [Fig. 5(b)]. In this regime the coupling of the antenna with the cavity is so large that the rectangular resonator also displays features typical of pseudointegrable billiards [34]. In the case of L-shaped cavities the pseudointegrability is connected with the shape of the resonator and the influence of the antenna for the results is smaller. This explains smaller changes of the level spacing distribution of L-shaped resonators with frequency.

Measured spectra are characterized by the parameter λ obtained in the process of fitting the distribution (8) to the data analyzed. Figure 6 shows the experimental results for the frequency range $10 \text{ GHz} < \nu < 15 \text{ GHz}$. Values of the parameter λ are equal to 0.62, 0.68, and 0.75 for the frequency ranges limited by 10, 15, and 18 GHz, respectively. The confidence level Q was of the order of $10^{-6} - 10^{-7}$ in each case. Corresponding values for the rectangular resonators are 0.19, 0.34, and 0.58. Comparison of these numbers allows us to conclude that (a) the spectra of L-shaped resonators are closer to the GOE than the spectra of rectangular resonators, (b) the properties of the spectra of L-shaped resonators depend only weakly on the frequency, and (c) due to presence of the antenna in the experimental apparatus, the spectral properties of rectangular and L-shaped resonators become similar for large frequencies.

V. CONCLUDING REMARKS

Numerical investigations of pseudointegrable quantum billiards allow us to conclude that the statistical properties of the spectra are neither described by the Poisson distribution nor the Wigner distribution. In contrast to integrable systems the analyzed spectra display a clear level repulsion. On the other hand, its character is different in comparison with the classically chaotic systems. This result suggests that the statistical properties

of quantized spectra are a good measure to judge, whether or not the underlying classical dynamics is chaotic and described by a positive Lyapunov exponent.

None of the known families of the distributions interpolating between Poisson and the GOE distribution can be considered as exact for the pseudointegrable systems. Random-matrix distribution (8) can be used, however, to parametrize the properties of the spectrum. Parameter λ might be interpreted as a measure of an effective coupling of the "chaotic" contribution to the integrable system due to presence of the singularities in the classically regular geodesic flow.

Level statistics and the spectral rigidity of pseudointegrable billiards are dependent on the genus of the manifold corresponding to the classical phase space. However, based on all the numerical data collected, we cannot state that statistical properties of quantum spectra are determined uniquely by the genus g . Further analysis of other polygonal billiards is needed to find definite evidence proving this hypothesis.

Numerical results concerning the properties of spectra of pseudointegrable quantum billiards are additionally supported by results of a microwave experiment with L-shaped cavities. There exist two independent sources of pseudointegrability in this case: the $3\pi/2$ corner of the microwave resonator and the wire antenna being a part of

the measurement device. The influence of the antenna increases with the frequency of microwaves. The observed frequency dependence of the λ parameters for the microwave L-shaped resonators shows that the effective genus increases with the frequency from 2 to 3. An analogous behavior was also found for the rectangular resonators [34]. The experiments show that the statistical properties of spectra of L-shaped resonators do not depend on the frequency so strongly as the spectra of the rectangular resonator.

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